ON THE DEGREE OF CONVERGENCE OF LAPLACE'S SERIES*

BY

T. H. GRONWALL

§ 1. Introduction.

Let S denote a sphere of unit radius, and refer its points to a geographical system of coördinates, θ being the polar distance and φ the longitude; the north pole and the zero meridian may be fixed arbitrarily. Further let $f(\theta, \varphi)$ be a uniform function of the position on the sphere, and suppose this function to be absolutely integrable in the Riemann sense, so that

$$\int_S |f(\theta, \varphi)|\, d\sigma$$

exists, where $d\sigma = \sin\theta\, d\theta\, d\varphi$ is the surface element of S. We finally denote by θ', φ' any point on S, the corresponding surface element being $d\sigma'$, and by γ the angle between the two vectors from the center of the sphere to the points θ, φ and θ', φ' respectively (or, in other words, the distance between the two points in question, measured on the great circle joining them). This angle γ is completely determined by the condition $0 \gtreqless \gamma \gtreqless \pi$, and we have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi' - \varphi).$$

Then *the Laplace series corresponding to* $f(\theta, \varphi)$ is

$$(1) \qquad f(\theta, \varphi) \sim \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_S f(\theta', \varphi')\, P_n(\cos\gamma)\, d\sigma',$$

P_n being the nth Legendre polynomial; the equivalence sign $\sim$ sign fies that the series is considered in a purely formal way, and implies no statement in regard to its convergence.

An important particular case is obtained by supposing $f(\theta, \varphi)$ independent of φ; writing $\cos\theta = x$ and $f(\theta, \varphi) = f(\cos\theta) = f(x)$, the Laplace series (1) becomes *the Legendre series corresponding to* $f(x)$:

$$(2) \qquad f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{2n+1}{2}\int_{-1}^{+1} f(y)\, P_n(y)\, dy.$$

* Presented to the Society February 22, 1913.

The present paper is concerned with the degree of approximation obtained by replacing $f(\theta, \varphi)$ by the sum of the first $n + 1$ terms in its Laplace series, or in other words, with the investigation of the function

$$(3) \quad r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \sum_{\nu=0}^{n} \frac{2\nu+1}{4\pi} \int_S f(\theta', \varphi') P_\nu(\cos \gamma)\, d\sigma'$$

under the assumption that for any two points θ, φ and θ', φ' the function $f(\theta, \varphi)$ satisfies a generalized Lipschitz condition of the form

$$(4) \qquad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma),$$

where, for γ increasing, $\omega(\gamma)$ never decreases, and $\omega(\gamma)/\gamma$ never increases, while $\lim_{\gamma=0} \omega(\gamma) = 0$. On account of the last condition, $f(\theta, \varphi)$ is obviously continuous all over the sphere S.

In a recent paper, Jackson* has treated the corresponding problem for the Legendre series (2), assuming that $f(x)$ satisfies the condition

$$(5) \quad |f(x_2) - f(x_1)| \leqq \omega(|x_2 - x_1|), \quad -1 \leqq x_1 \leqq 1, \quad -1 \leqq x_2 \leqq 1,$$

the function ω being subject to the same restrictions as above. He confines his investigation, however, to the points of the interval

$$(6) \qquad -1 + \epsilon \leqq x \leqq 1 - \epsilon,$$

where ϵ is any *positive* quantity. The methods of the present paper apply, on convenient specialization, to Legendre's series for $x = \pm 1$; whenever the occasion arises, we shall compare our results with those of Jackson and thus bring out the fundamental difference between the behavior of the Legendre series in the interior points of the interval $(-1, 1)$ and in the end points.

Section 2 of the present paper gives a lemma of fundamental importance: supposing $f(\theta, \varphi)$ in (1) limited by the condition that $|f(\theta, \varphi)| \leqq 1$ at every point of S, then there exists a function $f(\theta, \varphi)$ of this kind which makes the absolute value of the sum of the first $n + 1$ terms in Laplace's series a maximum ρ_n at a given point θ, φ, and it is shown that these constants ρ_n, which we shall call the Lebesgue constants for Laplace's series,† tend towards infinity with n as $\sqrt{n}$,‡ or more accurately, that §

* Dunham Jackson, I. *On the Degree of Convergence of the Development of a Continuous Function According to Legendre's Polynomials*, these Transactions, vol. 13 (1912), pp. 305–318.

† Lebesgue first showed the fundamental importance of the corresponding constants for Fourier's series; see *Leçons sur les séries trigonométriques*, Paris, 1906, § 45.

‡ Jackson shows (l. c., pp. 306–309) that the corresponding maximum for Legendre's series in the interval (6) is inferior to a constant multiple of $\log n$.

§ The first proof of this relation may be found in § 2 of my paper *Über die Laplace'sche Reihe* (Mathematische Annalen, vol. 74 (1913), pp. 213–270); the present proof contains several simplifications and also some new developments which will be used in § 4.

$$\lim_{n=\infty} \frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}.$$

In § 3 it will be shown that $f(\theta, \varphi)$ may be approximated by a sum $T_n(\theta, \varphi)$ of spherical harmonics of degree not exceeding n in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \leqq K\omega\left(\frac{1}{n}\right) \qquad (n = 1, 2, 3, \cdots)$$

at every point of the sphere S, K being a constant independent of n. The method of proof is closely related to that of Jackson* for the corresponding problem regarding $f(x)$.

In § 4, the preceding results are used to show that for all points of S

$$|r_n\{f(\theta, \varphi)\}| \leqq K'\omega\left(\frac{1}{n}\right)\sqrt{n} \qquad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n, and that, conversely, if $\Omega(\gamma)$ is a function subject to the same restrictions as $\omega(\gamma)$ and furthermore

$$\lim_{\gamma=0} \frac{\Omega(\gamma)}{\omega(\gamma)} = 0,$$

then there exists a function $f(\theta, \varphi)$ satisfying (4) and such that at a given point θ_0, φ_0 the inequality

$$|r_n\{f(\theta_0, \varphi_0)\}| \geqq K''\Omega\left(\frac{1}{n}\right)\sqrt{n}$$

is satisfied for an infinite number of values of n, the constant K'' being independent of n.†

§ 2. The Lebesgue constants for Laplace's series.

The sum of the first $n + 1$ terms in (1) is equal to

$$s_n\{f(\theta, \varphi)\} = \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\omega'$$

where

(7)
$$s_n(x) = \sum_{\nu=0}^{n} (2\nu + 1) P_\nu(x).$$

Limiting $f(\theta, \varphi)$ to the class of absolutely integrable functions which are not

* Dunham Jackson, II. *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Inaugural-Dissertation, Göttingen, 1911. III. *On Approximation by Trigonometric Sums and Polynomials*, these Transactions, vol. 13 (1912), pp. 491–515.

† For $f(x)$ and x limited by (6), $\sqrt{n}$ may be replaced by $\log n$ and $\Omega(1/n)$ by $\omega(1/n)$ in the two statements regarding r_n. (Jackson, I, theorems 1 and 3.)

greater than unity in absolute value at every point of S, we see at once that

$$|s_n\{f(\theta,\varphi)\}| \overline{\overline{<}} \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma',$$

and the absolute maximum

$$\rho_n(\theta,\varphi) = \max.\,|s_n\{f(\theta,\varphi)\}| = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma'$$

is reached at the point θ, φ by defining f so that

$$f(\theta',\varphi') = \operatorname{sgn.} s_n(\cos\gamma),$$

where, in the notation of Kronecker,

$$\operatorname{sgn.} a = \begin{cases} +1, & a > 0 \\ \ \ 0, & a = 0 \\ -1, & a < 0. \end{cases}$$

Moving the north pole of our coördinate system to the point θ, φ, we obtain $\gamma = \theta'$ and

$$\rho_n(\theta,\varphi) = \frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\,d\sigma' = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} |s_n(\cos\theta')|\sin\theta'\,d\theta'\,d\varphi'$$
$$= \tfrac{1}{2}\int_0^{\pi} |s_n(\cos\theta')|\sin\theta'\,d\theta',$$

so that $\rho_n(\theta,\varphi) = \rho_n$ is independent of θ, φ; writing $\cos\theta' = x$, we find

$$\rho_n = \tfrac{1}{2}\int_{-1}^{+1} |s_n(x)|\,dx, \qquad (n = 0, 1, 2, \cdots). \tag{8}$$

In order to investigate the order of magnitude of ρ_n with respect to n, we begin by developing an asymptotic expression for $s_n(\cos\theta)$, which will furnish approximations to the roots of the equation $s_n(\cos\theta) = 0$.

From the well-known equation of definition of Legendre's polynomials

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)\,z^n, \qquad |z| < 1, \tag{9}$$

where the square root equals unity for $z = 0$, the equally well-known formula

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1)\,P_n(x)\,z^n$$

is easily obtained, and on multiplying both sides by

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and using (7), it is seen at once that

$$\text{(10)} \qquad \frac{1+z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} s_n(x)\, z^n, \qquad |z| < 1.$$

If we write $1/z$ instead of z and introduce $x = \cos\theta$, this becomes

$$\text{(11)} \qquad \frac{z^2(z+1)}{(z-e^{\theta i})^{3/2}(z-e^{-\theta i})^{3/2}} = \sum_{n=0}^{\infty} s_n(\cos\theta)\, z^{-n}, \qquad |z| > 1,$$

the radicals being determined so that their real parts tend toward $+\infty$ when z tends toward infinity by real and positive values. Then the application of Cauchy's theorem gives

$$\text{(11)} \qquad s_n(\cos\theta) = \frac{1}{2\pi i}\int_C \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}\,dz,$$

the contour of integration C enclosing both of the singular points $z = e^{\theta i}$ and $z = e^{-\theta i}$. Now let ϵ be a positive quantity which will later be made to tend toward zero, and let C consist of the circuit C_1 around $z = e^{\theta i}$, composed of the three parts: (1) the straight line from $z = 0$ to $z = (1-\epsilon)e^{\theta i}$, (2) the circle $z = e^{\theta i} + \epsilon e^{\phi i}$, $\theta - \pi \leqq \phi \leqq \theta + \pi$, (3) the straight line from $z = (1-\epsilon)e^{\theta i}$ to $z = 0$, followed by the circuit C_2 around $z = e^{-\theta i}$, composed of the three parts: (4) the straight line from $z = 0$ to $z = (1-\epsilon)e^{-\theta i}$, (5) the circle $z = e^{-\theta i} + \epsilon e^{\phi i}$, $-\theta - \pi \leqq \phi \leqq \pi - \theta$, (6) the straight line from $z = (1-\epsilon)\,e^{-\theta i}$ to $z = 0$.

To make the passage to the limit $\epsilon = 0$ possible, (11) must be transformed so that the integrand in (11) becomes infinite of an order less than one at the singular points, and to this purpose we use the identity

$$\frac{z+1}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}} = -\frac{1}{z-1}\left[\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} + 2z\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\right]$$

which gives, upon integration by parts,

$$\begin{aligned}\int \frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}\,dz = &-\int \frac{z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ &-2\int \frac{z^{n+2}}{(z-1)}\frac{d}{dz}\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ = &-\frac{2z^{n+2}}{z-1}\cdot\frac{1}{(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}} \\ &+\int \frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz \\ &-\int \frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}\,dz.\end{aligned}$$

Integrate over the circuit C_1; the expression outside of the integral signs vanishes at the beginning and the end of C_1 ($z = 0$); making $\epsilon = 0$, the integrals over (2) vanish, and those over (3) become equal to the corresponding ones over (1), owing to the fact that $(z - e^{\theta i})^{\frac{1}{2}}$ changes its sign when z describes the circle (2). We therefore obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz = \frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{(2n+3)z^{n+1}}{(z-1)(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz$$

$$-\frac{1}{\pi i}\int_0^{e^{\theta i}}\frac{2z^{n+2}}{(z-1)^2(z-e^{\theta i})^{\frac{1}{2}}(z-e^{-\theta i})^{\frac{1}{2}}}dz,$$

the path of integration being the straight line from $z = 0$ to $z = e^{\theta i}$, and the initial values of the radicals for $z = 0$ having their real parts positive (being positive for $z = +\infty$ and never passing through zero as z moves along the real axis). We now introduce a new variable of integration, making

$$z = e^{\theta i}(1-u), \qquad 0 \leqq u \leqq 1;$$

then, on account of the manner of fixing the values of the radicals for $z = 0$,

$$(z - e^{\theta i})^{\frac{1}{2}} = e^{\frac{\theta-\pi}{2}i}\cdot u^{\frac{1}{2}}, \qquad u^{\frac{1}{2}} \geqq 0,$$

$$(z - e^{-\theta i})^{\frac{1}{2}} = \sqrt{2\sin\theta}\cdot e^{\frac{\pi}{4}i}\left(1 - \frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}},$$

where

$$\left(1 - \frac{ue^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}\right)^{\frac{1}{2}} = +1 \quad \text{for} \quad u = 0.$$

Introducing all this in the integrals above, and writing

$$\alpha = \frac{e^{\frac{\theta-\pi}{2}}}{2\sin\frac{\theta}{2}}, \qquad \beta = \frac{e^{\left(\theta-\frac{\pi}{2}\right)i}}{2\sin\theta}, \tag{12}$$

we obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{z^{n+1}(z+1)}{(z-e^{\theta i})^{\frac{3}{2}}(z-e^{-\theta i})^{\frac{3}{2}}}dz = \frac{2n+3}{\pi}\cdot\frac{e^{(n+1)\theta i-\frac{3}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}}du$$

$$-\frac{2}{\pi}\cdot\frac{e^{(n+\frac{3}{2})\theta i-\frac{1}{4}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2(2\sin\theta)^{\frac{1}{2}}}\int_0^1\frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}}du.$$

In exactly the same way it is seen that the corresponding integral over the circuit C_2 has a value which is conjugate to the one just obtained for C_1, so that, by (11),

$$
\begin{aligned}
s_n(\cos\theta) &= \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{1}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \\
&\quad - \frac{4}{\pi} \boldsymbol{R} \frac{e^{(n+\frac{3}{2})\theta i - \frac{1}{4}\pi i}}{\left(2\sin\frac{\theta}{2}\right)^2 (2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} du,
\end{aligned}
\tag{13}
$$

where $\boldsymbol{R}$ denotes the real part of the expression following it.

This formula will now easily give the desired asymptotic expression for $s_n(\cos\theta)$. On account of (12), we have for $0 \leqq u \leqq 1$

$$
|1-\alpha u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\frac{\theta}{2} \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},
$$

$$
|1-\beta u|^2 = \left(1-\frac{u}{2}\right)^2 + \frac{u^2}{4}\cot^2\theta \geqq \left(1-\frac{u}{2}\right)^2 \geqq \frac{1}{4},
$$

so that

$$
\frac{1}{|1-\alpha u|} \leqq 2, \qquad \frac{1}{|1-\beta u|} \leqq 2; \tag{14}
$$

furthermore the identity

$$
\frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1 = \int_0^u \left(\frac{\alpha}{(1-\alpha u)^2(1-\beta u)^{\frac{1}{2}}} + \frac{1}{2}\cdot\frac{\beta}{(1-\alpha u)(1-\beta u)^{\frac{3}{2}}} \right) du
$$

shows, by the aid of (14), that

$$
\left| \frac{1}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} - 1 \right| < \int_0^u (2^{\frac{5}{2}}|\alpha| + 2^{\frac{3}{2}}|\beta|) du = \left(\frac{2^{\frac{5}{2}}}{2\sin\frac{\theta}{2}} + \frac{2^{\frac{3}{2}}}{2\sin\theta} \right) u < \frac{8u}{\sin\theta}.
$$

From the first integral in (13) we now obtain

$$
\begin{aligned}
&\left| \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{1}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \right. \\
&\qquad \left. - \frac{2(2n+3)}{\pi} \boldsymbol{R} \frac{e^{(n+1)\theta i - \frac{1}{4}\pi i}}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(1-u)^{n+1} du \right| \\
&\qquad < \frac{2(2n+3)}{\pi} \cdot \frac{1}{2\sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \cdot \frac{8}{\sin\theta} \int_0^1 u^{\frac{1}{2}}(1-u)^{n+1} du,
\end{aligned}
$$

or evaluating the Euler integrals of the first kind occurring in this formula and reducing the result by means of $\Gamma(x+1) = x\,\Gamma(x)$,

$$\left| \frac{2(2n+3)}{\pi} R \frac{e^{(n+1)\theta i - \frac{1}{4}\pi i}}{2 \sin\frac{\theta}{2}(2 \sin \theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+1}}{(1-\alpha u)(1-\beta u)^{\frac{1}{2}}} du \right.$$

$$\left. - \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \cdot \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2 \sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \right|$$

$$< \frac{4}{\pi}\left(n+\frac{3}{2}\right) \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{7}{2}\right)} \cdot \frac{16}{2 \sin\frac{\theta}{2}(2 \sin\theta)^{\frac{1}{2}}}$$

$$< \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{8}{2 \sin\frac{\theta}{2}(2 \sin\theta)^{\frac{1}{2}}}.$$

The second integral in (13) may be estimated directly by (14), which gives

$$\left| \frac{4}{\pi} R \frac{e^{(n+\frac{3}{2})\theta i - \pi i}}{\left(2 \sin\frac{\theta}{2}\right)^2 (2 \sin\theta)^{\frac{1}{2}}} \int_0^1 \frac{u^{-\frac{1}{2}}(1-u)^{n+2}}{(1-\alpha u)^2 (1-\beta u)^{\frac{1}{2}}} du \right|$$

$$< \frac{4}{\pi} \cdot \frac{2^2 \cdot 2^{\frac{1}{2}}}{\left(2 \sin\frac{\theta}{2}\right)^2 (2 \sin\theta)^{\frac{1}{2}}} \int_0^1 u^{-\frac{1}{2}}(i-u)^{n+2} du$$

$$= \frac{4}{\pi} \cdot \frac{2^{\frac{5}{2}}}{\left(2 \sin\frac{\theta}{2}\right)^2 (2 \sin\theta)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \cdot (n+2) \cdot \Gamma(n+2)}{\left(n+\frac{5}{2}\right)\Gamma\left(n+\frac{5}{2}\right)}$$

$$< \frac{4}{\pi} \cdot \frac{2 \cdot 2}{2 \sin\frac{\theta}{2}(2 \sin\theta)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)}.$$

Introducing these approximations in (13), we obtain the asymptotic expression

$$(15) \qquad \begin{aligned} s_n(\cos\theta) = {} & \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)} \frac{\sin\left((n+1)\theta - \frac{\pi}{4}\right)}{2 \sin\frac{\theta}{2}(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{4}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{5}{2}\right)} \cdot \frac{24\,\eta(n,\theta)}{2 \sin\frac{\theta}{2}(2 \sin\theta)^{\frac{1}{2}}}, \end{aligned}$$

where

$$|\eta(n,\theta)| < 1 \qquad \text{for} \qquad 0 \leqq \theta \leqq \pi \qquad \text{and} \qquad n = 1, 2, 3, \cdots.$$

To obtain some information regarding the zeros of $s_n(\cos\theta)$, we write (15) in the form

$$\text{(16)} \qquad \frac{\pi}{8}\,\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)} \cdot 2\sin\frac{\theta}{2}\,(2\sin\theta)^{\frac{1}{2}} \cdot s_n(\cos\theta) = \sin\theta\sin\left((n+1)\,\theta - \frac{\pi}{4}\right) + \frac{24\eta(n,\theta)}{2n+3}.$$

Subjecting the integers n and ν to the inequalities

$$\text{(17)} \qquad n \geqq 4^5, \qquad (n+1)^{\frac{3}{4}} \leqq \nu \leqq n - (n+1)^{\frac{3}{4}},$$

we introduce the special value of θ

$$\text{(18)} \qquad \theta_0 = \frac{4\nu+1}{n+1}\,\frac{\pi}{4} + \frac{\kappa\pi}{(n+1)^{\frac{1}{4}}}, \qquad \kappa = \pm 1,$$

in the expression (16). From (18) it follows that

$$(n+1)\,\theta_0 - \frac{\pi}{4} = \nu\pi + \frac{\kappa\pi}{(n+1)^{\frac{1}{4}}},$$

and

$$\sin\left((n+1)\,\theta_0 - \frac{\pi}{4}\right) = (-1)^\nu \kappa \sin\frac{\pi}{(n+1)^{\frac{1}{4}}}.$$

Furthermore, we have, by (17) and (18),

$$0 < \frac{\nu\pi}{n+1} < \theta_0 < \frac{\nu+1}{n+1}\pi < \pi,$$

and, on account of $\sin x > 2x/\pi$ for $0 < x < \pi$, we find, in case $0 < \theta_0 \leqq \pi/2$,

$$\sin\theta_0 > \sin\frac{\nu\pi}{n+1} > \frac{2}{\pi}\cdot\frac{\nu\pi}{n+1} \geqq \frac{2}{(n+1)^{\frac{1}{4}}},$$

and in case $\pi/2 < \theta_0 < \pi$,

$$\sin\theta_0 > \sin\frac{\nu+1}{n+1}\pi = \sin\frac{n-\nu}{n+1}\pi > \frac{2}{\pi}\cdot\frac{n-\nu}{n+1}\pi \geqq \frac{2}{(n+1)^{\frac{1}{4}}},$$

and finally

$$\left|\sin\left((n+1)\,\theta_0 - \frac{\pi}{4}\right)\right| = \sin\frac{\pi}{(n+1)^{\frac{1}{4}}} > \frac{2}{\pi}\cdot\frac{\pi}{(n+1)^{\frac{1}{4}}} = \frac{2}{(n+1)^{\frac{1}{4}}}.$$

From these inequalities it follows that

$$\left|\sin\theta_0 \sin\left((n+1)\,\theta_0 - \frac{\pi}{4}\right)\right| > \frac{2}{(n+1)^{\frac14}} \cdot \frac{2}{(n+1)^{\frac14}}$$

$$= \frac{4}{(n+1)^{\frac12}} > \frac{12}{n+1} > \frac{24}{2n+3} > \left|\frac{24\eta\,(n,\theta_0)}{2n+3}\right|.$$

and consequently, for $\theta = \theta_0$, the expression (16) has the same sign as the first term on its right side, so that

$$\text{sgn.}\, s_n(\theta_0) = (-1)^{\nu} \cdot \kappa,$$

whence we conclude that $s_n(\theta)$ changes sign an odd number of times in the interval

$$\frac{4\nu+1}{n+1}\,\frac{\pi}{4} - \frac{\pi}{(n+1)^{\frac54}} < \theta < \frac{4\nu+1}{n+1}\,\frac{\pi}{4} + \frac{\pi}{(n+1)^{\frac54}}, \tag{19}$$

whenever n and ν satisfy the inequalities (17). In case the interval (19) should contain several zeros of odd order ot $s_n(\theta)$, let θ_ν be the smallest among them; we then obviously have

$$\theta_\nu = \frac{4\nu+1}{n+1}\,\frac{\pi}{4} + \frac{h_\nu \pi}{(n+1)^{\frac54}}, \qquad |h_\nu| < 1. \tag{20}$$

Returning to the expression (8), let $x_1, x_2, \cdots, x_m$, where $1 > x_1 > x_2 > \cdots > x_m > -1$, be the points at which $s_n(x)$ changes sign in the interval $-1 \overline{\gtrless} x \overline{\gtrless} 1$. Such points exist, on account of (20), and on the other hand, $s_n(x)$ being, by definition, a polynomial of degree n, we have $m \overline{\gtrless} n$.* For reasons of symmetry, we also write $x_0 = 1$ and $x_{m+1} = -1$; as $P_n(1) = 1$, $s_n(1)$ is positive, and consequently

$$|s_n(x)| = (-1)^{\lambda} s_n(x), \qquad x_\lambda \leqq x \leqq x_{\lambda+1}, \qquad \lambda = 0, 1, \cdots, m,$$

so that

$$\rho_n = \frac12\int_{-1}^{1} |s_n(x)|\,dx = \left(\frac12\int_{x_1}^{x_0} + \frac12\int_{x_2}^{x_1} + \cdots + \frac12\int_{x_{m+1}}^{x_m}\right)|s_n(x)|\,dx$$

$$= \frac12\sum_{\lambda=0}^{m}(-1)^{\lambda}\int_{x_{\lambda+1}}^{x_\lambda} s_n(x)\,dx.$$

* We do not need to use the fact that $m = n$, which is easily proved by Christoffel's formula

$$s_n(x) = (n+1)\,\frac{P_n(x) - P_{n+1}(x)}{1-x}$$

and the fact that the zeros of $P_n(x)$ lie between -1 and $+1$ and separate those of $P_{n+1}(x)$. Substituting for x two consecutive roots of $P_{n+1}(x) = 0$, we obtain values of $s_n(x)$ of opposite signs, and consequently the zeros of $s_n(x)$ separate those of $P_{n+1}(x)$, or $m = n$.

Using the formula

$$s_n(x) = \frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}, \tag{21}$$

we obtain

$$\rho_n = \frac{1}{2}\sum_{\lambda=0}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda) - P_n(x_{\lambda+1}) - P_{n+1}(x_{\lambda+1})],$$

or rearranging the terms,

$$\rho_n = \frac{1}{2}[P_n(x_0) + P_{n+1}(x_0)] + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]$$
$$+ \frac{(-1)^{m+1}}{2}[P_n(x_{m+1}) + P_{n+1}(x_{m+1})],$$

or finally, observing that $P_n(x_0) = 1$, $P_n(x_{m+1}) = (-1)^n$,

$$\rho_n = 1 + \sum_{\lambda=1}^{m}(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]. \tag{22}$$

Let ν_1 be the smallest, and ν_2 the greatest, of the integers ν satisfying (17); then to each ν, where $\nu_1 \lesseqgtr \nu \lesseqgtr \nu_2$, there corresponds, according to (20), an index λ_ν $(1 \leqq \lambda_\nu \leqq m)$ such that

$$\cos\theta_\nu = x_{\lambda_\nu}, \qquad \nu_1 \leqq \nu \leqq \nu_2.$$

We obviously have $\lambda_{\nu+1} > \lambda_\nu$, and by the definition of θ_ν, there is an even number of roots of $s_n(\cos\theta) = 0$ (or none) between θ_ν and $\theta_{\nu+1}$, so that

$$\lambda_{\nu+1} - \lambda_\nu \equiv 1 \pmod{2},$$

or

$$\lambda_{\nu+1} - (\nu + 1) \equiv \lambda_\nu - \nu \pmod{2}.$$

We may therefore write (22) in the form

$$\rho_n = (-1)^{\lambda_{\nu_1} - \nu_1}\sum_{\nu=\nu_1}^{\nu_2}(-1)^\nu[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] + 1$$
$$+ \sum_\lambda{}'(-1)^\lambda[(P_n(x_\lambda) + P_{n+1}(x_\lambda)],$$

the second sum extending over such values of λ as are not of the form λ_ν, $(\nu_1 \leqq \nu \leqq \nu_2)$. As $m \leqq n$, the number of terms in Σ' does not exceed $n - (\nu_2 - \nu_1 + 1)$, which is less than or equal to $2(n+1)^{\frac{3}{4}}$, by (17). Each term in Σ' not exceeding $|P_n(x_\lambda)| + |P_{n+1}(x_\lambda)| < 1 + 1 = 2$ in absolute value, we have *

$$\left|1 + \sum_\lambda{}'(-1)^\lambda[P_n(x_\lambda) + P_{n+1}(x_\lambda)]\right| < 1 + 2(n+1)^{\frac{3}{4}} \cdot 2 = O(n^{\frac{3}{4}}),$$

* The notation $f(n) = O(g(n))$ signifies that for n sufficiently large, $|f(n)| < Ag(n)$, where A is a constant independent of n. Similarly, $f(n) = o(g(n))$ stands for $\lim\limits_{n=\infty}\frac{f(n)}{g(n)} = 0$.

and consequently

$$\rho_n = (-1)^{\lambda_{\nu_1}-\nu_1} \sum_{\nu=\nu_1}^{\nu_2} (-1)^{\nu} \left[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu) \right] + O(n^{\frac{1}{2}}). \tag{23}$$

Our next step will obviously be the asymptotic determination of the general term in the series above, and to this purpose we use the following formula, due to Stieltjes* and constituting a generalization of the well-known asymptotic formula of Laplace:

$$\begin{aligned} P_n(\cos\theta) = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg[& \frac{\cos\left(\frac{2n+1}{2}\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{1^2}{2(2n+3)} \cdot \frac{\cos\left(\frac{2n+3}{2} - \frac{3\pi}{4}\right)}{(2\sin\theta)^{\frac{3}{2}}} + \cdots \\ & + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2p-3)^2}{2\cdot 4 \cdots (2p-2)(2n+3)\cdots(2n+2p-1)} \frac{\cos\left(\frac{2n+2p-1}{2}\theta - \frac{2p-1}{4}\pi\right)}{(2\sin\theta)^{\frac{2p-1}{2}}} \\ & + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2p-1)^2}{2\cdot 4 \cdots 2p(2n+3)(2n+5)\cdots(2n+2p+1)} \frac{M(p,n,\theta)}{(2\sin\theta)^{\frac{2p+1}{2}}} \Bigg], \end{aligned} \tag{24}$$

where

$$|M(p,n,\theta)| < 2 \quad \text{for} \quad 0 \leqq \theta \leqq \pi.$$

For our purpose, it is sufficiently accurate to make $p = 1$ in (24), and we obtain

$$\begin{aligned} P_n(\cos\theta) + P_{n+1}(\cos\theta) = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \frac{2\cos\frac{\theta}{2}\cos\left((n+1)\theta - \frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}} \\ & + \frac{1}{(2n+3)(2\sin\theta)^{\frac{3}{2}}} \Bigg[-2\sin\theta\cos\left(\frac{2n+3}{2}\theta - \frac{\pi}{4}\right) \\ & + \frac{M(1,n,\theta)}{2} + \frac{2n+2}{2n+5} \frac{M(1,n+1,\theta)}{2} \Bigg] \Bigg\} \\ = \frac{2}{\pi} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \Bigg\{ & \cos\left((n+1)\theta - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta}{2}} \\ & + \frac{M'(n,\theta)}{(2n+3)(2\sin\theta)^{\frac{3}{2}}} \Bigg\}, \end{aligned} \tag{25}$$

* T. J. Stieltjes, *Sur les polynômes de Legendre*, Annales de la Faculté des Sciences de Toulouse, ser. I, vol. 4 (1890), pp. G1 – G17. The method used above for arriving at the asymptotic expression for $s_n(\cos\theta)$ was suggested by Stieltjes' proof of (24).

where

$$| M'(n,\theta) | < 4 \quad \text{for} \quad 0 \leqq \theta \leqq \pi .$$

By (20) and the various inequalities following (18), we have

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right) = (-1)^\nu \cos\frac{h_\nu \pi}{(n+1)^{\frac{1}{8}}}$$

$$= (-1)^\nu\left[1 + O\left(\frac{h_\nu^2\pi^2}{(n+1)^{\frac{1}{4}}}\right)\right] = (-1)^\nu\left[1 + O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right],$$

$$\cot\frac{\theta_\nu}{2} < \frac{1}{\sin\frac{\theta_\nu}{2}} < \frac{1}{\frac{2}{\pi}\cdot\frac{\nu\pi}{2(n+1)}} \leqq (n+1)^{\frac{3}{8}};$$

consequently

$$\cos\left((n+1)\theta_\nu - \frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} = (-1)^\nu\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{16}}}\right),$$

and furthermore

$$\frac{M'(n,\theta_\nu)}{(2n+3)(2\sin\theta_\nu)^{\frac{3}{2}}} = O\left(\frac{1}{2n+3}\cdot(n+1)^{\frac{9}{16}}\right) = O\left(\frac{1}{n^{\frac{1}{16}}}\right);$$

finally Stirling's formula,

$$\log\Gamma(x+1) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \sum_{\nu=1}^{p}\frac{(-1)^{\nu-1}B_\nu}{(2\nu-1)\cdot 2\nu}\cdot\frac{1}{x^{2\nu-1}}$$

$$+ \frac{(-1)^p B_{p+1}}{(2p+1)(2p+2)}\,\frac{\vartheta}{x^{2p+1}}, \qquad 0 < \vartheta < 1,$$

gives, for $p = 1$, and since $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\frac{2}{\pi}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} = \frac{2}{\sqrt{\pi n}}\left[1 - \frac{1}{4n} + O\left(\frac{1}{n^2}\right)\right] \tag{26}$$

Introducing all this in (25), we obtain

$$(-1)^\nu\,[P_n(\cos\theta_\nu) + P_{n+1}(\cos\theta_\nu)] = \frac{2}{\sqrt{\pi n}}\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{9}{16}}}\right),$$

whence, by (23),

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + \sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{9}{16}}}\right) + O(n^{\frac{7}{16}}).$$

In the second sum, each term is less than a constant multiple of $n^{-\frac{9}{16}}$, and the number of terms is less than n, so that

$$\sum_{\nu=\nu_1}^{\nu_2} O\left(\frac{1}{n^{\frac{9}{16}}}\right) = O\left(n\cdot\frac{1}{n^{\frac{9}{16}}}\right) = O(n^{\frac{7}{16}}),$$

and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1}\rho_n = \frac{2}{\sqrt{\pi n}}\sum_{\nu=\nu_1}^{\nu_2}\sqrt{\cot\frac{\theta_\nu}{2}} + O(n^{\frac{3}{8}}),$$

which may be written

$$(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\cdot\frac{n+1}{n}\sum_{\nu=\nu_1}^{\nu_2}\left(\frac{4\nu+3}{n+1}\frac{\pi}{4} - \frac{4\nu-1}{n+1}\frac{\pi}{4}\right)\sqrt{\cot\frac{\theta_\nu}{2}} + O\left(\frac{1}{n^{\frac{1}{16}}}\right),$$

By (17) and (20), we have

$$\frac{4\nu+3}{n+1}\frac{\pi}{4} > \theta_\nu > \frac{4\nu-1}{n+1}\frac{\pi}{4},$$

and the preceding equation gives, when we increase n indefinitely,*

$$\lim_{n=\infty}(-1)^{\lambda_{\nu_1}-\nu_1}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

From the equation of definition (8), it follows that ρ_n is positive, and consequently

$$(-1)^{\lambda_{\nu_1}-\nu_1} = 1$$

for all sufficiently large values of n, so that

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = \frac{2}{\pi\sqrt{\pi}}\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta.$$

Making the substitution $\cot\theta/2 = u^2$, we obtain

$$\int_0^\pi\sqrt{\cot\frac{\theta}{2}}\,d\theta = \int_0^\infty\frac{4u^2\,du}{1+u^4} = \int_{-\infty}^{+\infty}\frac{2u^2\,du}{1+u^4},$$

and by a classical application of Cauchy's theorem, the latter integral equals $2\pi i$ times the sum of the residues of the integrand in the upper half plane, or

$$2\pi i\left\{\left(\frac{2u^2}{4u^3}\right)_{u=\frac{1+i}{\sqrt{2}}} + \left(\frac{2u^2}{4u^3}\right)_{u=\frac{-1+i}{\sqrt{2}}}\right\} = \pi i\sqrt{2}\left(\frac{1}{1+i} + \frac{1}{-1+i}\right) = \pi\sqrt{2},$$

so that finally,

$$\lim_{n=\infty}\frac{\rho_n}{\sqrt{n}} = 2\sqrt{\frac{2}{\pi}}. \tag{27}$$

§ 3. Approximation to functions satisfying a generalized Lipschitz condition by finite sums of spherical harmonics.

THEOREM I. *Let $f(\theta,\varphi)$ be a uniform function of the position on the sphere S, satisfying the generalized Lipschitz condition for any two points θ,φ and θ',φ':*

$$|f(\theta',\varphi') - f(\theta,\varphi)| \leqq \omega(\gamma), \tag{28}$$

* The integrand decreasing monotonously as θ increases, this passage to the limit is seen at once to be legitimate, although the integral obtained is an improper one.

*the function $\omega(\gamma)$ being subject to the conditions**

$$\omega(\gamma) \leqq \omega(\gamma'), \qquad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad for \quad 0 < \gamma \leqq \gamma' < \pi; \tag{29}$$
$$\lim_{\gamma=0} \omega(\gamma) = 0.$$

Then there exists, for every positive integer n, a sum $T_n(\theta, \varphi)$, of a finite number of spherical harmonics of order $\overline{\overline{<}}\, n$, approximating $f(\theta, \varphi)$ in such a way that

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| \overline{\overline{<}}\, K\omega(1/n)$$

at every point of the sphere S, K being a constant independent of n.

In order to prove this theorem, we shall use a special type of functions $T_n(\theta, \varphi)$, defined by†

$$T_n(\theta, \varphi) = \frac{\displaystyle\int_S f(\theta', \varphi') \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}, \tag{30}$$

where the integer m is related to n by

$$2(m-1) \overline{\overline{<}}\, n < 2m. \tag{31}$$

The function $T_n(\theta, \varphi)$ is actually a sum of a finite number of spherical harmonics of order $\leqq n$; to prove this assertion, we start from the relation

$$\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}} = \frac{e^{\frac{m\gamma i}{2}} - e^{-\frac{m\gamma i}{2}}}{e^{\frac{\gamma i}{2}} - e^{-\frac{\gamma i}{2}}} = e^{-(m-1)\frac{\gamma i}{2}} \cdot \frac{1 - e^{m\gamma i}}{1 - e^{\gamma i}} = e^{-(m-1)\frac{\gamma i}{2}} \sum_{\nu=0}^{m-1} e^{\nu\gamma i}, \tag{32}$$

whence

$$\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = e^{-2(m-1)\gamma i} \sum_{\nu_1=0}^{m-1} \sum_{\nu_2=0}^{m-1} \sum_{\nu_3=0}^{m-1} \sum_{\nu_4=0}^{m-1} e^{(\nu_1+\nu_2+\nu_3+\nu_4)\gamma i} = \sum_{\lambda=-2(m-1)}^{+2(m-1)} c_\lambda e^{\lambda\gamma i}.$$

Changing γ and λ into $-\gamma$ and $-\lambda$, we see at once that $c_\lambda = c_{-\lambda}$ and the

* From the last condition, it follows that $f(\theta, \varphi)$ is continuous in every point of S.

† The corresponding finite trigonometric sums for a function $f(x)$ satisfying the ordinary Lipschitz condition $|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|$, $\lambda =$ const., were first introduced by Jackson (see papers II and III, quoted in § 1).

last equation becomes

$$(33)\qquad \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 = c_0 + \sum_{\lambda=1}^{2(m-1)} c_\lambda (e^{\lambda\gamma i} + e^{-\lambda\gamma i}) = c_0 + 2\sum_{\lambda=1}^{2(m-1)} c_\lambda \cos\lambda\gamma .$$

Expressing $\cos\lambda\gamma$ as a polynomial of degree λ in $\cos\gamma$, and this polynomial being expressed by the Legendre polynomials in $\cos\gamma$ of degree $\leqq \lambda$, we reduce (30) to the form

$$T_n(\theta, \varphi) = \sum_{\nu=0}^{2(m-1)} a_\nu \int_S f(\theta', \varphi') P_\nu(\cos\gamma)\, d\sigma',$$

the general term on the right side being obviously a spherical harmonic of order $\nu \overline{\gtrless} 2(m-1) \overline{\gtrless} n$.

Having established this point, we may write (30) in the form

$$f(\theta, \varphi) - T_n(\theta, \varphi) = \frac{\displaystyle\int_S [f(\theta, \varphi) - f(\theta', \varphi')] \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'},$$

and by (28) we obtain

$$(34)\qquad |f(\theta, \varphi) - T_n(\theta, \varphi)| \overline{\gtrless} \frac{\displaystyle\int_S \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}{\displaystyle\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma'}$$

In the integrals, we now move the north pole to the point θ, φ; then $\gamma = \theta'$ and $d\sigma' = \sin\gamma\, d\gamma d\varphi$, so that

$$\int_S \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = \int_0^\pi \int_0^{2\pi} \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma\, d\varphi$$

$$= 2\pi \int_0^\pi \omega(\gamma) \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma ,$$

$$\int_S \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 d\sigma' = 2\pi\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

By the aid of (29), we obtain

$$\int_0^\pi \omega(\gamma)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma = \int_0^{1/n} + \int_{1/n}^\pi \overline{\gtrless} \int_0^{1/n} \omega\left(\frac{1}{n}\right)\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$+ \int_{1/n}^\pi \frac{\omega\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + n\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

$$< \omega\left(\frac{1}{n}\right)\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma + 2m\omega\left(\frac{1}{n}\right)\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma.$$

nce $n < 2m$, by (31), it follows from (34) that

$$(35)\qquad |f(\theta,\varphi) - T_n(\theta,\varphi)| < \omega\left(\frac{1}{n}\right)\left[1 + \frac{2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}{\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma}\right]$$

Observing that $\sin x > 2x/\pi$ for $0 < x < \pi/2$ and $\sin x < x$ for $0 < x < \pi$, we find

$$2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma < 2m\int_0^\pi \gamma\left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{\pi}}\right]^4 \gamma\, d\gamma = 2\pi^4 m\int_0^\pi \frac{\sin^4\frac{m\gamma}{2}}{\gamma^2}\, d\gamma$$

$$= \pi^4 m^2\int_0^{\frac{m\pi}{2}} \frac{\sin^4\gamma}{\gamma^2}\, d\gamma < \pi^4 m^2\int_0^\infty \frac{\sin^4\gamma}{\gamma^2}\, d\gamma,$$

$$\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma > \int_0^{\frac{\pi}{2}} \left[\frac{\sin\frac{u\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma > \int_0^{\frac{\pi}{2}} \left[\frac{\sin\frac{m\gamma}{2}}{\frac{\gamma}{2}}\right]^4 \cdot \frac{2}{\pi}\gamma\, d\gamma$$

$$= \frac{32}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^4\frac{m\gamma}{2}}{\gamma^3}\, d\gamma = \frac{8}{\pi} m^2 \int_0^{\frac{m\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma \geqq \frac{8}{\pi} m^2 \int_0^{\frac{\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma,$$

and from (34) we obtain

$$|f(\theta,\varphi) - T_n(\theta,\varphi)| < \omega\left(\frac{1}{n}\right)\left\{1 + \frac{\pi^5 \int_0^\infty \frac{\sin^4\gamma}{\gamma^2}\, d\gamma}{8\int_0^{\frac{\pi}{4}} \frac{\sin^4\gamma}{\gamma^3}\, d\gamma}\right\} \quad (n = 1, 2, 3, \cdots),$$

which proves our theorem.

Although not necessary for the application in § 4, it is interesting to derive from (35) a fairly small numerical upper limit for the constant K in theorem I. With the notations

$$I_m = \frac{1}{4m^2}\int_0^\pi \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma, \qquad I'_m = \frac{1}{2m}\int_0^\pi \gamma \left[\frac{\sin\frac{m\gamma}{2}}{\sin\frac{\gamma}{2}}\right]^4 \sin\gamma\, d\gamma$$

equation (35) may be written

$$|f(\theta,\varphi) - T_n(\theta,\varphi)| < \left(1 + \frac{I'_m}{I_m}\right)\omega\left(\frac{1}{n}\right), \tag{36}$$

and making $\gamma = 2u$ and using the inequality $u\cos u < \sin u \left(0 < u < \frac{\pi}{2}\right)$, we have

$$\begin{aligned} I_m &= \frac{1}{m^2}\int_0^{\frac{\pi}{2}} \frac{\sin^4 mu \cos u}{\sin^3 u}\, du, \\ I'_m &= \frac{4}{m}\int_0^{\frac{\pi}{2}} \frac{u\sin^4 mu\cos u}{\sin^3 u}\, du < \frac{4}{m}\int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u}\, du. \end{aligned} \tag{37}$$

Considering first the integral I_m, we integrate by parts and obtain

$$m^2 I_m = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^4 mu \cdot d\left(\frac{1}{\sin^2 u}\right) = -\frac{1}{2}\sin^4\frac{m\pi}{2} + 2m\int_0^{\frac{\pi}{2}} \frac{\sin^3 mu\cos mu}{\sin^2 u}\, du$$

$$= \frac{(-1)^m - 1}{4} - 2m\int_0^{\frac{\pi}{2}} \sin^3 mu\cos mu \cdot d\cot u$$

$$= \frac{(-1)^m - 1}{4} + 2m^2\int_0^{\frac{\pi}{2}} (3\sin^2 mu\cos^2 mu - \sin^4 mu)\cot u\, du;$$

as $3 \sin^2 mu \cos^2 mu - \sin^4 mu = \frac{1}{2} (\cos 2\, mu - \cos 4mu)$, this gives

$$I_m = \frac{(-1)^m - 1}{4m^2} + j_m,$$

where

$$j_m = \int_0^{\frac{\pi}{2}} \cot u \, (\cos 2\, mu - \cos 4mu) \, du.$$

Consider the difference $j_m - j_{m-1}$; the identities

$$\begin{aligned}
&\cot u [\cos 2mu - \cos 4mu - (\cos 2(m-1)u - \cos 4(m-1)u)] \\
&\quad = \cot u [-2 \sin u \sin (2m-1)u + 2 \sin 2u \sin (4m-2)u] \\
&\quad = -2 \cos u \sin (2m-1)u + 4 \cos^2 u \sin (4m-2)u \\
&\quad = -2 \cos u \sin (2m-1)u + 2(1 + \cos 2u) \sin (4m-2)u \\
&\quad = -\sin 2mu - \sin (2m-2)u + 2 \sin (4m-2)u + \sin 4mu \\
&\qquad\qquad + \sin (4m-4)u
\end{aligned}$$

give, upon integration between the limits 0 and $\pi/2$,

$$j_m - j_{m-1} = \frac{1}{2m}[(-1)^m - 1] + \frac{1}{2m-2}[(-1)^{m-1} - 1] + \frac{2}{2m-1},$$

where, for $m = 1$, the second term on the right side should be replaced by zero. On account of $j_0 = 0$, we conclude that

$$\begin{aligned}
j_m &= \sum_{\nu=1}^{m} \left[\frac{1}{2\nu}[(-1)^\nu - 1] + \frac{1}{2\nu - 2}[(-1)^{\nu-1} - 1] + \frac{2}{2\nu - 1} \right] \\
&= \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \sum_{\nu=1}^{m-1} \left[\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu - 1} \right],
\end{aligned}$$

the result being also true for $m = 1$, if the sum on the right side is replaced by zero. It is now seen that

$$(38) \quad I_m = \sum_{\nu=1}^{m-1} \left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu - 1} \right) + \frac{(-1)^m - 1}{2m} + \frac{2}{2m-1} + \frac{(-1)^m - 1}{4m^2},$$

and

$$I_{m+1} - I_m = \frac{(-1)^m - 1}{2m} - \frac{(-1)^m - 1}{4m^2} - \frac{(-1)^m + 1}{2(m+1)} + \frac{2}{2m+1} - \frac{(-1)^m + 1}{4(m+1)^2}$$

or, when the cases m odd and m even are separated,

$$I_{m+1} - I_m = \begin{cases} \dfrac{1}{2m^2(2m+1)}, & m \text{ odd}, \\[2ex] \dfrac{1}{2(m+1)^2(2m+1)}, & m \text{ even}, \end{cases}$$

so that in both cases

(39) $$I_{m+1} > I_m .$$

For $m = 2$, (38) gives the value $2/3$, whence by (39)

(40) $$I_m \gtreqq \tfrac{2}{3} = 0.6667 - \quad \text{for} \quad m \gtreqq 2 .$$

To form an idea of the approximation furnished by (40), we observe that by equations (39) and (38),

$$\begin{aligned} I_m < I_{2m} < \lim_{m=\infty} I_{2m} &= \lim_{m=\infty} \sum_{\nu=1}^{2m-1} \left(\frac{(-1)^\nu - 1}{\nu} + \frac{2}{2\nu - 1} \right) \\ &= \lim_{m=\infty} \left(\sum_{\nu=1}^{2m-1} \frac{2}{2\nu-1} - \sum_{\nu=1}^{m} \frac{2}{2\nu-1} \right) = \lim_{m=\infty} \sum_{\nu=m+1}^{2m-1} \frac{2}{2\nu-1} \\ &= \lim_{m=\infty} \frac{1}{2m-1} \sum_{\nu=1}^{m-1} \frac{1}{\frac{1}{2} + \frac{\nu}{2m-1}} = \int_{\frac{1}{2}}^{1} \frac{dx}{x} \\ &= \log 2 = 0.6931 + . \end{aligned}$$

Passing to the last integral in (37), we have

$$\frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u} du = \frac{2}{m} \int_0^{\frac{\pi}{2}} (1 - \cos 2mu) \left(\frac{\sin mu}{\sin u} \right)^2 du .$$

Writing $\gamma = 2u$ in (32) and squaring, we obtain

$$\left(\frac{\sin mu}{\sin u} \right)^2 = e^{-2(m-1)ui} \sum_{\nu_1=0}^{m-1} \sum_{\nu_2=0}^{m-1} e^{2(\nu_1+\nu_2)ui} = \sum_{\lambda=-(m-1)}^{+(m-1)} c'_\lambda e^{2\lambda ui};$$

changing u and λ into $-u$ and $-\lambda$, we find $c'_\lambda = c'_{-\lambda}$, so that

$$\left(\frac{\sin mu}{\sin u} \right)^2 = c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u ,$$

whence

$$\begin{aligned} \frac{4}{m} \int_0^{\frac{\pi}{2}} \frac{\sin^4 mu}{\sin^2 u} du &= \frac{2}{m} \int_0^{\frac{\pi}{2}} (1 - \cos 2mu) \left(c'_0 + 2 \sum_{\lambda=1}^{m-1} c'_\lambda \cos 2\lambda u \right) du \\ &= \frac{2}{m} \cdot \frac{\pi}{2} c'_0 . \end{aligned}$$

From (32) it is seen immediately that c'_0 is the constant term in the expansion of

$$x^{-(m-1)} \left(\frac{1 - x^m}{1 - x} \right)^2 = x^{-(m-1)} (1 - 2x^m + x^{2m}) \sum_{\nu=0}^{\infty} (\nu + 1) x^\nu ,$$

or $c'_0 = m$, so that we obtain from (37)

(41) $$I'_m < \pi .$$

Equations (40) and (41) now give

$$1 + \frac{I'_m}{I_m} < 1 + \frac{3\pi}{2} < 6 \quad \text{for} \quad m \geqq 2;$$

that is, on account of (31), for $n \geqq 2$, and from (36) we derive the desired result

$$|f(\theta, \varphi) - T_n(\theta, \varphi)| < 6\omega\left(\frac{1}{n}\right) \quad (n = 2, 3, 4, \cdots). \tag{42}$$

Before leaving this subject, we shall evaluate the integral

$$J_m = \frac{1}{m^3}\int_0^{\frac{\pi}{2}} \left(\frac{\sin mu}{\sin u}\right)^4 du \tag{43}$$

occurring in Jackson's investigation of the approximation to a function $f(x)$ of period 2π and satisfying a Lipschitz condition

$$f(x_2) - f(x_1) | \leqq \lambda | x_2 - x_1 |$$

by finite trigonometric sums. By an asymptotic process, Jackson* shows that

$$J_m > J_{m+1}, \qquad \lim_{m=\infty} J_m = \frac{\pi}{3}. \tag{44}$$

Writing $\gamma = 2u$, we obtain at once from (33)

$$J_m = \frac{c_0}{m^3} \cdot \frac{\pi}{2}.$$

By (32) and (33), c_0 is obviously the constant term in the expansion of

$$x^{-2(m-1)}\left(\frac{1 - x^m}{1 - x}\right)^4 = x^{-2(m-1)}(1 - 4x^m + 6x^{2m} - 4x^{3m} + x^{4m}) \times \sum_{\nu=0}^{\infty} \frac{(\nu+1)(\nu+2)(\nu+3)}{1 \cdot 2 \cdot 3} x^\nu,$$

so that, the terms in the infinite series with $\nu = 2(m-1)$ and $\nu = m - 2$ being the only ones contributing to c_0,

$$c_0 = \frac{(2m-1) \cdot 2m(2m+1)}{1 \cdot 2 \cdot 3} - 4 \cdot \frac{(m-1) \cdot m(m+1)}{1 \cdot 2 \cdot 3} = \frac{m}{3}(2m^2 + 1),$$

and consequently

$$J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2}\right). \tag{45}$$

From this expression, the two properties (44) follow at once.

* In the paper previously quoted as III, pp. 503–507.

§ 4. The degree of convergence of Laplace's series for functions satisfying a generalized Lipschitz condition.

THEOREM II. *When $f(\theta, \varphi)$ satisfies the generalized Lipschitz condition*

$$|f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma) \tag{28}$$

for any two points θ, φ and θ', φ' on the sphere S, the function $\omega(\gamma)$ being subject to the conditions

$$\omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \text{ for } 0 < \gamma \leqq \gamma' < \pi, \ \lim_{\gamma=0} \omega(\gamma) = 0, \tag{29}$$

then $f(\theta, \varphi)$ is approximated by the first $n+1$ terms of its Laplace series in such a way that the remainder

$$r_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - s_n\{f(\theta, \varphi)\} = f(\theta, \varphi) - \frac{1}{4\pi}\int_S f(\theta', \varphi')\, s_n(\cos\gamma)\, d\sigma'$$

satisfies the inequality

$$|r_n\{f(\theta, \varphi)\}| \leqq K'\omega\left(\frac{1}{n}\right)\sqrt{n} \qquad (n = 1, 2, 3, \cdots),$$

K' being a constant independent of n.

The principle of the following proof is due to Lebesgue, in the case of Fourier's series.* From the identity

$$f(\theta, \varphi) = T_n(\theta, \varphi) + [f(\theta, \varphi) - T_n(\theta, \varphi)]$$

and the definition of the remainder r_n, it follows that

$$r_n\{f(\theta, \varphi)\} = r_n\{T_n(\theta, \varphi)\} + r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\};$$

the Laplace series of $T_n(\theta, \varphi)$ being obviously this function itself, it is seen that

$$r_n\{T_n(\theta, \varphi)\} = 0.$$

The definition of r_n then gives

$$|r_n\{f(\theta, \varphi)| = |r_n\{f(\theta, \varphi) - T_n(\theta, \varphi)\}| \leqq |f(\theta, \varphi) - T_n(\theta, \varphi)|$$
$$+ \frac{1}{4\pi}\int_S |f(\theta', \varphi') - T_n(\theta', \varphi')|\,|s_n(\cos\gamma)|\, d\sigma',$$

and if we apply theorem I,

$$|r_n\{f(\theta, \varphi)\}| \leqq K\omega\left(\frac{1}{n}\right) + K\omega\left(\frac{1}{n}\right)\cdot\frac{1}{4\pi}\int_S |s_n(\cos\gamma)|\, d\sigma'$$
$$= K\omega\left(\frac{1}{n}\right)(1 + \rho_n),$$

* H. Lebesgue, *Sur la représentation trigonométrique des fonctions, etc.*, Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; pp. 196–197; *Sur les intégrales singulières*, Annales de la Faculté des Sciences de Toulouse, ser. 3, vol. 1 (1910), pp. 25–117; pp. 116–117. Compare also Jackson, I, II, III.

by the definition of ρ_n at the beginning of § 2. On account of (27), there exists a constant c independent of n and such that

$$\rho_n \leqq c\sqrt{n} \qquad (n = 1, 2, 3, \cdots).$$

The preceding inequality now shows that

$$|r_n\{f(\theta, \varphi)\}| \leqq K(1+c\sqrt{n})\omega\left(\frac{1}{n}\right) \leqq K(1+c)\,\omega\left(\frac{1}{n}\right)\sqrt{n} = K'\omega\left(\frac{1}{n}\right)\sqrt{n},$$

which proves our theorem.

It follows, in particular, from theorem II that *whenever*

$$\lim_{n=\infty} \omega\left(\frac{1}{n}\right)\sqrt{n} = 0,$$

the Laplace series corresponding to $f(\theta, \varphi)$ *is convergent and equal to this function at every point of the sphere* S.

In the particular case of the Legendre series corresponding to a function $f(x) = f(\cos\theta)$ satisfying the condition (5), which is analogous to (28), Jackson* has shown that for any positive ϵ there exists a constant $K'(\epsilon)$ independent of n (but depending, of course, on ϵ) such that

$$|r_n\{f(x)\}| < K'(\epsilon)\,\omega\left(\frac{1}{n}\right)\log n \qquad (n = 2, 3, 4, \cdots),$$

uniformly in the interval

$$-1+\epsilon \leqq x \leqq 1-\epsilon.$$

This result cannot be extended to the limiting case $\epsilon = 0$; on the contrary, we have the following theorem, which may be considered as the reciprocal of theorem II:

THEOREM III. *Let* $\omega(\gamma)$ *be a function subject to the conditions*

$$(29) \quad \omega(\gamma) \leqq \omega(\gamma'), \quad \frac{\omega(\gamma)}{\gamma} \geqq \frac{\omega(\gamma')}{\gamma'} \quad \textit{for } 0 < \gamma \leqq \gamma' < \pi; \ \lim_{\gamma=0}\omega(\gamma) = 0,$$

and $\Omega(\gamma)$ *any other function such that*

$$(46) \qquad \lim_{\gamma=0}\frac{\Omega(\gamma)}{\omega(\gamma)} = 0;$$

there exists a function $f(\theta, \varphi)$ *satisfying the generalized Lipschitz condition*

$$(28) \qquad |f(\theta', \varphi') - f(\theta, \varphi)| \leqq \omega(\gamma)$$

for any two points θ, φ *and* θ', φ' *on the sphere* S, *and such that, at a given point*

* Paper referred to as I, theorem (1).

θ_0, φ_0, the remainder in its Laplace series satisfies the inequality

$$| r_n \{ f(\theta_0, \varphi_0) \} | \geqq K'' \Omega\left(\frac{1}{n}\right) \sqrt{n}$$

for an infinity of values of n, the constant K'' being independent of n.

To prove this theorem, we shall construct a special function $f(\theta, \varphi)$ having the required properties,* choosing the given point at $\theta_0 = 0$, $\varphi_0 = 0$, and making $f(\theta, \varphi) = F(\theta)$ independent of φ, so that our Laplace series will reduce to a Legendre series. Our function $F(\theta)$ will be built up from elements of the form $F_n(\theta)$, this latter function being defined by

$$(47) \quad F_n(\theta) = \sin\left((n+1)\theta - \frac{\pi}{4}\right) \quad \text{for} \quad \frac{1}{n+1}\frac{\pi}{4} \leqq \theta \leqq \frac{4\mu+1}{n+1}\frac{\pi}{4},$$
$$= 0 \text{ outside of this interval,}$$

μ and n being positive integers (or zero) to be determined later. $F_n(\theta)$ is continuous, as it vanishes at the end points of the interval above, which we may denote by (μ, n).

By (47), we have $F_n(\theta) = 0$ for $\theta = 0$, so that, developing $F_n(\theta)$ in a Laplace series, we find

$$r_n \{ F_n(0) \} = - s_n \{ F_n(0) \} = - \frac{1}{4\pi} \int_S F_n(\theta') s_n(\cos\gamma) \, d\sigma',$$

or, introducing $d\sigma' = \sin\theta' \, d\theta' \, d\varphi'$ and using the well-known formula

$$\int_0^{2\pi} P_n(\cos\gamma) \, d\varphi' = 2\pi P_n(\cos\theta) P_n(\cos\theta'),$$

making $\theta = 0$, so that $P_n(\cos\theta) = 1$, and finally writing θ instead of θ',

$$r_n \{ F_n(0) \} = - \frac{1}{2} \int_0^{\pi} F_n(\theta) s_n(\cos\theta) \sin\theta \, d\theta.$$

Since $F_n(\theta)$ is identically equal to zero in part of the interval of integration, we may write

$$r_n \{ F_n(0) \} = - \frac{1}{2} \int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} F_n(\theta) s_n(\cos\theta) \sin\theta \, d\theta;$$

or introducing the value $F_n(\theta) = \sin\left((n+1)\theta - \frac{\pi}{4}\right)$ in the interval in question, and using (15),

* In principle, the method of proof is due to Lebesgue, who applied it to Fourier's series; see papers by Lebesgue and Jackson previously quoted.

$$r_n\{F_n(0)\} = -\frac{2}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}\times$$

$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}\,(2\sin\theta)^{\frac{1}{2}}}\right]d\theta. \tag{48}$$

Now assume $\mu \leqq \frac{n}{2}$; we then have, in the interval of integration in (48)

$$\cos\frac{\theta}{2} > \frac{1}{\sqrt{2}}, \qquad \theta > \sin\theta > \frac{2}{\pi}\theta, \qquad 2\sin\frac{\theta}{2} > \frac{2}{\pi}\theta, \tag{49}$$

so that

$$\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}\,d\theta > \frac{1}{2}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

$$= \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\theta}}\,d\theta$$

$$> \frac{1}{2}\sum_{\nu=1}^{\mu}\int_{\frac{4\nu-3}{n+1}\frac{\pi}{4}}^{\frac{4\nu+1}{n+1}\frac{\pi}{4}}\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{\sqrt{\frac{4\nu+1}{n+1}\frac{\pi}{4}}}\,d\theta \tag{50}$$

$$= \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\sum_{\nu=1}^{\mu}\frac{1}{\sqrt{4\nu+1}}$$

$$> \frac{\sqrt{\pi}}{8}\cdot\frac{1}{\sqrt{n+1}}\int_1^{\mu+1}\frac{du}{\sqrt{4u+1}}$$

$$= \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right).$$

For the second part of the integral in (48) we obtain, using (49) and observing that $|\eta(n,\theta)| < 1$,

$$
\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}\,(2\sin\theta)^{\frac{1}{2}}}\,d\theta\right| < \int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{24}{2n+2}\cdot\frac{1}{\frac{2}{\pi}\theta\cdot\left(\frac{4}{\pi}\theta\right)^{\frac{1}{2}}}\,d\theta
$$

51)
$$
= \frac{3\pi^{\frac{3}{2}}}{n+1}\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \frac{d\theta}{\theta^{\frac{3}{2}}}
$$

$$
= \frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right);
$$

and on combining (50) with (51), it is seen that

$$
\left|\int_{\frac{1}{n+1}\frac{\pi}{4}}^{\frac{4\mu+1}{n+1}\frac{\pi}{4}} \left[\frac{\sin^2\left((n+1)\theta-\frac{\pi}{4}\right)}{(2\sin\theta)^{\frac{1}{2}}}\cos\frac{\theta}{2}+\frac{24\eta(n,\theta)}{2n+3}\,\frac{\sin\left((n+1)\theta-\frac{\pi}{4}\right)}{2\sin\frac{\theta}{2}\,(2\sin\theta)^{\frac{1}{2}}}\right]d\theta\right|
$$

$$
> \frac{\sqrt{\pi}}{16}\cdot\frac{1}{\sqrt{n+1}}\left(\sqrt{4(\mu+1)+1}-3\right)-\frac{12\pi}{\sqrt{n+1}}\left(1-\frac{1}{\sqrt{4\mu+1}}\right)
$$

$$
> c_1\sqrt{\frac{\mu}{n+1}}
$$

for n and μ sufficiently large, c_1, as well as $c_2, c_3, \cdots$ which will be introduced below, signifying a certain positive constant, independent of n and μ. By (26) we have, for n sufficiently large,

$$
\frac{2}{\pi}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} > c_2\sqrt{n+1},
$$

so that we finally obtain from (48)

$$
(52) \qquad |r_n\{F_n(0)\}| > c_3\sqrt{4\mu+5} \quad \text{for} \quad n \geqq n', \quad \mu \geqq \mu', \quad 2\mu \leqq n.
$$

On account of (27) there exists a c_4 such that

$$
(53) \qquad \rho_n < c_4\sqrt{n+1} \qquad (n = 1, 2, \cdots).
$$

We now select an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ subject to the conditions

$$
n_1 \geqq n',
$$

(54)
$$
\omega\left(\frac{\pi}{n_i+1}\right) < \frac{c_3}{4c_4}\cdot\frac{1}{\sqrt{n_{i-2}+1}}\cdot\omega\left(\frac{\pi}{n_{i-1}+1}\right),
$$

$$
\Omega\left(\frac{1}{n_i}\right) < \frac{1}{\sqrt{n_{i-1}+1}}\,\omega\left(\frac{1}{n}\right) \qquad (i = 3, 4, 5, \cdots).
$$

The determination of such a sequence is possible, because after determining $n_1, n_2, \cdots, n_{i-1}$, we may satisfy the last two inequalities (54) by making n_i large enough, on account of $\lim_{\gamma=0} \omega(\gamma) = 0$ and $\lim_{\gamma=0} \Omega(\gamma)/\omega(\gamma) = 0$.

To each n_i we now associate a μ_i by the condition

$$\frac{4\mu_i + 1}{n_i + 1} < \frac{1}{n_{i-1} + 1} \leqq \frac{4\mu_i + 5}{n_i + 1}; \tag{55}$$

then any two intervals (μ_i, n_i) and (μ_j, n_j), in which, by (47), $F_{n_i}(\theta)$ and $F_{n_j}(\theta)$ respectively are not identically zero, have no points in common. Finally, we make v_i equal to zero or unity, according as

$$\left| r_{n_i} \left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j \, \omega\left(\frac{\pi}{n_j + 1}\right) F_{n_j}(0) \right\} \right|$$

exceeds

$$\frac{c_3}{4\pi} \sqrt{\frac{n_i + 1}{n_{i-1} + 1}} \cdot \omega\left(\frac{\pi}{n_i + 1}\right)$$

or not. Then the expression

$$F(\theta) = \sum_{i=1}^{\infty} \frac{1}{2\pi} v_i \, \omega\left(\frac{\pi}{n_i + 1}\right) F_{n_i}(\theta) \tag{56}$$

has the properties required by theorem III.

Since for any given value of θ not more than one of the terms in the series (56) is different from zero (the intervals in which any two different functions $F_{n_i}(\theta)$ are not identically zero, having no point in common), the series is convergent. Furthermore, $F(\theta)$ satisfies the condition (28). To prove this, we first assume that θ and θ' belong to the same interval (μ_i, n_i); in this interval we have

$$F(\theta) = \frac{v_i}{2\pi} \omega\left(\frac{\pi}{n_i + 1}\right) \sin\left((n_i + 1)\theta - \frac{\pi}{4}\right).$$

We now determine an integer λ by the condition

$$\theta + \frac{2\lambda\pi}{n_i + 1} \leqq \theta' < \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

and denote by θ'' that one of the two quantities

$$\theta + \frac{2\lambda\pi}{n_i + 1} \qquad \text{and} \qquad \theta + \frac{2\lambda + 2}{n_i + 1}\pi$$

which is nearer to θ'; then we have

$$\left| \theta'' - \theta' \right| \leqq \frac{\pi}{n_i + 1}, \tag{57}$$

$$F(\theta') - F(\theta) = F(\theta') - F(\theta'')$$
$$= \frac{v_i}{2\pi}\,\omega\left(\frac{\pi}{n_i+1}\right)\cdot(n_i+1)(\theta'-\theta'')\cos\left((n_i+1)\,\xi-\frac{\pi}{4}\right),$$

ξ lying between θ' and θ'', and consequently

$$|F(\theta') - F(\theta)| < \frac{1}{2}\cdot\frac{\omega\left(\dfrac{\pi}{n_i+1}\right)}{\dfrac{\pi}{n_i+1}}\,|\theta''-\theta'|$$
$$= \frac{1}{2}\cdot\frac{\omega\left(\dfrac{\pi}{n_i+1}\right)}{\dfrac{\pi}{n_i+1}}\cdot\frac{|\theta''-\theta'|}{\omega(|\theta''-\theta'|)}\cdot\omega(|\theta''-\theta'|);$$

or, on account of (57) and the fact that $\omega(\gamma)/\gamma$ does not increase with γ,

$$(58)\qquad |F(\theta') - F(\theta)| < \tfrac{1}{2}\,\omega(|\theta''-\theta'|) \leqq \tfrac{1}{2}\,\omega(|\theta'-\theta|).$$

Next suppose that θ belongs to the interval (μ_i, n_i) and θ' to the interval (μ_j, n_j), and let θ_i be the end point of (μ_i, n_i) nearest to θ', and θ_j the end point of (μ_j, n_j) nearest to θ_i; then, since $F(\theta_i) = F(\theta_j) = 0$, we have

$$|F(\theta') - F(\theta)| = |F(\theta') - F(\theta_j) + F(\theta_j) - F(\theta_i) + F(\theta_i) - F(\theta)|$$
$$\leqq |F(\theta') - F(\theta_j)| + |F(\theta_i) - F(\theta)|.$$

Applying (58) and considering that $\omega(\gamma)$ does not decrease as γ increases, we find

$$|F(\theta') - F(\theta_j)| < \tfrac{1}{2}\,\omega(|\theta'-\theta_j|) \leqq \tfrac{1}{2}\,\omega(|\theta'-\theta|),$$
$$|F(\theta_i) - F(\theta)| < \tfrac{1}{2}\,\omega(|\theta_i-\theta|) \leqq \tfrac{1}{2}\,\omega(|\theta'-\theta|),$$

so that finally, for any values of θ and θ' in the interval $(0, \pi)$

$$|F(\theta') - F(\theta)| < \omega(|\theta'-\theta|).$$

Now $|\theta'-\theta| \leqq \gamma$, since $|\theta'-\theta|$ is one of the sides in the right-angled spherical triangle of which γ is the hypothenuse, and consequently

$$|F(\theta') - F(\theta)| < \omega(\gamma).$$

We now complete the proof by showing that

$$|r_{n_i}\{F(0)\}| \geqq K''\,\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i} \qquad (i = 1, 2, 3, \cdots).$$

On account of (52) and (55), we have

$$\left|r_{n_i}\left\{\frac{1}{2\pi}\,\omega\left(\frac{\pi}{n_i+1}\right)F_{n_i}(0)\right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that in case $v_i = 1$ we have, in view of the definition of v_i,

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0)\right\}\right| \geqq \left| r_{n_i}\left\{ \frac{1}{2\pi}\omega\left(\frac{\pi}{n_i+1}\right) F_{n_i}(0)\right\}\right|$$

$$- \left| r_{n_i}\left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0)\right\}\right| > \frac{c_3}{2\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right)$$

$$- \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) = \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_{i+1}+1}\right).$$

In case $v_i = 0$, the definition of v_i gives

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0)\right\}\right| = \left| r_{n_i}\left\{ \sum_{j=1}^{i-1} \frac{1}{2\pi} v_j \omega\left(\frac{\pi}{n_j+1}\right) F_{n}(0)\right\}\right|$$

$$> \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right),$$

so that we always have

$$\left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(0)\right\}\right| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right). \tag{59}$$

From (56) and (59) it follows that

$$\begin{aligned} \left| r_{n_i}\left\{ F(\theta)\right\}\right| &\geqq \left| r_{n_i}\left\{ \sum_{j=1}^{i} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta)\right\}\right| \\ &\quad - \left| r_{n_i}\left\{ \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta)\right\}\right|, \end{aligned} \tag{60}$$

and since

$$\left| \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta)\right| \leqq \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)$$

for all values of θ, it follows from the definition of ρ_n and the inequality (53) that

$$\begin{aligned} &\left| r_{n_i}\left\{ \sum_{j=i+1}^{\infty} \frac{1}{2\pi} v_j\, \omega\left(\frac{\pi}{n_j+1}\right) F_{n_j}(\theta)\right\}\right| \\ &\leqq \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot \frac{1}{4\pi}\int_S |s_{n_i}(\cos\gamma)|\, d\sigma' = \frac{1}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\cdot \rho_{n_i} \\ &< \frac{c_4}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}. \end{aligned} \tag{61}$$

Making $\theta = 0$, we conclude from (60), (59) and (61) that

$$|r_{n_i}\{F(0)\}| > \frac{c_3}{4\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) - \frac{c_4}{2\pi}\omega\left(\frac{\pi}{n_{i+1}+1}\right)\sqrt{n_i+1}.$$

The second inequality in (54) shows that the right-hand expression is greater than

$$\frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{\pi}{n_i+1}\right) \geqq \frac{c_3}{8\pi}\sqrt{\frac{n_i+1}{n_{i-1}+1}}\,\omega\left(\frac{1}{n_i}\right),$$

since

$$\frac{\pi}{n_i+1} \geqq \frac{1}{n_i},$$

and from the last inequality in (54) we finally obtain

$$\left| r_{n_i}\left\{F(0)\right\}\right| > \frac{c_3}{8\pi}\,\Omega\left(\frac{1}{n_i}\right)\sqrt{n_i+1} \qquad (i=1,\,2,\,3,\,\cdots)$$

which completes the proof of our theorem.

In the case of the Legendre series, Jackson has shown (l. c., I, theorem III) that there exists a K'' independent of n and a function $f(x) = f(\cos\theta)$ satisfying (5) such that

$$|\, r_n\{f(0)\}\,| > K''\,\omega\left(\frac{1}{n}\right)\log n$$

for an infinity of values of n. Apart from the introduction of $\sqrt{n}$ instead of $\log n$ necessitated by the asymptotic expression for ρ_n, our theorem III differs from Jackson's result in two respects: first, by the introduction of the function $\Omega(\gamma)$ converging toward zero with γ more strongly than $\omega(\gamma)$, and second, by the fact that while the two conditions

$$|\, f(\cos\theta') - f(\cos\theta)\,| \overline{\gtrless}\ \omega(\,|\cos\theta' - \cos\theta|\,) \qquad \text{(Jackson)},$$

$$|\, f(\cos\theta') - f(\cos\theta)\,| \overline{\gtrless}\ \omega(\,|\theta' - \theta|\,) \qquad \text{(present paper)}$$

are essentially equivalent in the interior of $0 < \theta < \pi$, this is no longer the case in the vicinity of $\theta = 0$, since

$$\lim_{\theta,\,\theta'=0}\frac{\cos\theta' - \cos\theta}{\theta' - \theta} = 0.$$

CHICAGO, ILL.,
February 10, 1913.